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Bounds on Manipulation by Merging in Weighted Voting Games

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Abstract: *Manipulation by merging* in weighted voting games is a voluntary action of would-be *strategic agents* who come together to form a bloc in anticipation of receiving more payoff over the outcomes of games. Previous works have identified manipulation by merging in weighted voting games as a problem. This is because the increase in payoff or power (depending on the settings under consideration) that may be achieved by strategic agents in a game is at the deficit of other agents who are being denied some utilities that are due to them. Thus, the inability to limit (or understand) the effects of this manipulation may undermine the confidence agents have in decisions made via weighted voting games. If the results are not seen as fair, agents may refuse to abide by decisions made in this manner. We propose two non-trivial bounds to characterize the effects of this menace using the well-known *Banzhaf* power index. The two bounds are also *within constant factors*.

1 INTRODUCTION

Autonomous agents in complex environments may need to work together to achieve desired goals. This is an important feature of many multiagent environments where individual agents lack all the required capabilities, skills, and knowledge to complete tasks alone. Agents may thus resort to *cooperation* such as coalition formation, to complete tasks while being compensated with payoffs. One way of modeling such cooperation is via *weighted voting games* (WVGs). See for example, (Chalkiadakis et al., 2012). In a WVG, a *quota* is given and each agent has an associated weight. A subset of agents whose total weight meets or exceeds the quota is said to be *winning*. Agents' power in such games is measured using *power indices*. The *Banzhaf* power index (Penrose, 1946; Banzhaf, 1965) is one of the best-known indices for measuring agents' power in WVGs. The power of an agent reflects its ability to influence or affect the outcomes of decision-making processes.

Even though WVGs are useful in modeling cooperation among players for making joint decisions, they are not immune from the vulnerability of manipulation (i.e., dishonest behavior) by some players called *manipulators*, or referred to as being *strategic*, that may be present in the games. With the possibility of manipulation, it becomes difficult to establish or maintain trust in such games. This problem of insincere and manipulative behaviors among agents in WVGs has received attention of many researchers in recent years. See the work of (Bachrach and Elkind, 2008; Aziz et al., 2011; Lasisi and Allan, 2012; Lasisi and Allan, 2013; Lasisi and Allan, 2014; Rey and Rothe, 2014). Manipulation by merging involves voluntary coordinated action of strategic agents who come together to form a *bloc* by merging their weights into a single weight (Felsenthal and Machover, 2002; Aziz et al., 2011). The agents in the bloc are assumed to be assimilated voters since they can no more vote as individual voters in the new game, rather as a bloc. The new game consists of the previous agents in the original game that are not assimilated, as well as the bloc formed by the assimilated voters.

Strategic agents merge their weights in anticipation of gaining more power over the outcomes of games. In a beneficial merge, merged agents are compensated commensurate with their share of the power gained by the bloc. Agents in the bloc are assumed to be working cooperatively and have transferable utility. Thus, proceeds can easily be distributed among the manipulators without bickering. Common settings that may be vulnerable to such attack are online elections, rating systems, electronic negotiation, and auctions. See for example, (Yokoo et al., 2004), where the effects of false-name bids in combinatorial auction as a form of Internet fraud was studied.

A motivation for this problem can be found in decision-making, e.g., in negotiation settings. Consider a set of agents, $A = \{a_1, a_2, \ldots, a_n\}$, negotiating on how to allocate some budgets, B. Let a payoff method, such as the Banzhaf index, allocate B

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as, say, $P = \{p_1, p_2, \ldots, p_n\}$, to agents, A, respectively, based on their weights. Suppose some strategic agents, $S \subset A$, merge their weights to form a single bloc, they may be able to increase their share of the budget.

We motivate this problem further using a real-world example from the social choice domain. Consider a parliament consisting of five political parties, A, B, C, D, and E, which have 20, 30, 40, 50, and, 50 representatives (i.e., weights), respectively. This parliament is to vote on a \$100 million spending bill and how much of this amount should be controlled by each party. Furthermore, the bill requires a quota, $q \in [111, 120]$, i.e., the number of votes to pass. Assuming that all members of a political party votes in the same direction on a bill, the Banzhaf index allocates the amount of the spending bill to be controlled by each party as follows: $A =$ \$4m, $B =$ \$20m, $C =$ \$20m, $D =$ \$28m, and $E =$ \$28m. Now, suppose political parties A, B, and D, merge their weights to form a bloc with weight 100. The sum of thes initial allocation to each party in the bloc is \$4m + \$20m + \$28m = \$52m. However, the new allocation of the amount by Banzhaf index to the manipulators' bloc in the altered game is \$60m, which is more than \$52m.

The scenario described above obviously raises the following important questions that we seek to answer in this research work: *What is the extent of budgets, payoffs, or power (depending on the settings under consideration) that manipulators may gain? Analogously, what is the amount of damage that is caused to the non-manipulating agents in the games?*

This research is thus primarily motivated by the need to provide insights into understanding the details of the problem of this insincere and manipulative behavior in WVGs. We are concern that the inability to limit (or understand) the effects of this manipulation may undermine the confidence agents have in decisions made via WVGs. If the results from this decision-making process are not seen as fair, agents may refuse to abide by decisions made in this manner. We propose two new and non-trivial theoretical bounds to characterize the effects of manipulation by merging in WVGs using the well-known *Banzhaf* power index. The proposed bounds are for the case when the number of strategic agents in the g ames is 2. The bounds are also found to be within constant factors. Our results complement those of a previous work, (Lasisi and Lasisi, 2015), which propose two tight bounds (upper and lower) for this problem, also for the case when the number of strategic agents is 2, but using the Shapley-Shubik index.

2 RELATED WORK

WVGs are widely studied (Brams, 1975; Felsenthal and Machover, 1998; Taylor and Zwicker, 1999; Laruelle, 1999; Matsui and Matsui, 2000). They have found applications in many real-world environments, including the United Nations Security Council, the Electoral College of the United States, the IMF (Leech, 2002; Alonso-Meijide and Bowles, 2005), the Council of Ministers, and the European Community (Felsenthal and Machover, 1998).

The issue of WVGs design has also recently received attention of many researchers in the field (Aziz et al., 2007; Fatima et al., 2008; de Keijzer et al., 2010). The *Shapley value* (Shapley, 1953), its variant, *Shapley-Shubik* index (Shapley and Shubik, 1954), and the *Banzhaf* index (Penrose, 1946; Banzhaf, 1965) are the best-known power indices used in measuring power of agents in WVGs. Other lesser known power indices are Deegan-Packel (Deegan and Packel, 1978), Johnston (Johnston, 1978), and Holler-Packel (Holler and Packel, 1983) indices.

WVGs are vulnerable to various forms of dishonest behaviors, referred to as *manipulations*. These manipulations are due to strategic players that may be present in the games. Prominent among these forms of behaviors are manipulations by *splitting* and *merging* (Bachrach and Elkind, 2008; Lasisi and Allan, 2010; Aziz et al., 2011; Lasisi and Allan, 2012; Lasisi and Allan, 2013; Lasisi and Allan, 2014). Unlike in merging where two or more strategic agents merge their weights to form a single bloc, manipulation by splitting involves a strategic agent splitting its weight among two or more false agents in anticipation of gaining more power. Note that these two forms of manipulation have received attention of many researchers for the cases when the number, k, of strategic agents involved is either $k = 2$ or $k > 2$. However, none of these work considered the bounds on the extent of power that strategic agents may gain when they merge their weights using the Banzhaf power index. It is important to also point out that the effects of manipulation by splitting are well studied. See Table 1 for a summary on the state of the arts on manipulation by splitting in WVGs.

Previous work (Aziz et al., 2011) has shown that the problem of finding beneficial merge is NP-hard for the Banzhaf index. This complexity result seems sufficient to discourage would-be strategic agents from merging. We argue in the contrary that, NP-hardness result is a worst case measure, and only shows that at least an instance of the problem requires such complexity. Thus, the real life instances of WVGs that we care about may be easy to manipulate (Lasisi and Allan, 2013). Furthermore, Felsenthal and

Table 1: Summary of bounds for manipulation by splitting in weighted voting games.

Bounds	# Strategic agents	Shapely-Shubik Index	Banzhaf Index
Upper	$k = 2$	Bachrach & Elkind '08	Aziz & Peterson '09
	$k > 2$	Lasisi & Allan '14	Lasisi & Allan '14
Lower	$k = 2$	Bachrach & Elkind '08	Aziz et al. '11
	$k > 2$	Lasisi & Allan '14	Lasisi & Allan '14

Machover (Felsenthal and Machover, 2002) characterize situations when it is advantageous or disadvantageous for agents to merge, and show that using the Penrose-Banzhaf measure, merging can be advantageous or disadvantageous. Also, Lasisi and Allan (Lasisi and Allan, 2011) consider empirical evaluation of the extent of susceptibility of three power indices, namely, Shapley-Shubik, Banzhaf, and Deegan-Packel, to merging. Their results show that the three indices are susceptible to merging. However, none of these works provide bounds on the extent of power that manipulators may gain in the case that a merging is advantageous for the Banzhaf index.

Recently, Lasisi and Lasisi (Lasisi and Lasisi, 2015) propose two new tight bounds on the extent of power that manipulators may gain when they merge in WVGs. Their work uses the Shapley-Shubik index to characterize the effects of this problem, and considers the case when there are $k = 2$ strategic agents in the WVGs. The need for the characterization of the effects of this menace using the Banzhaf index was left, among others, as an open problem. We resolve this open problem by proposing two new and non-trivial theoretical bounds on the extent of power that strategic agents may gain with respect to manipulation by merging using the well-known Banzhaf power index to compute agents' power.

3 PRELIMINARIES

We present some preliminaries in this section, including definitions and notation, formal problem definition, and illustrative examples needed to provide necessary backgrounds.

3.1 Definitions and Notation

Let $I = \{1, \ldots, n\}$ be a set of $n \in \mathbb{N}$ agents. The non-empty subsets, $S \subseteq I$, are called *coalitions*.

Definition 1. *Simple Game*

A simple game is a coalitional game, (I, v), where $v : 2^I \to \{0, 1\}$. A coalition $S \subseteq I$ *wins* if $v(S) = 1$ and *loses* if $v(S) = 0$.

Definition 2. *Weighted Voting Game*

A weighted voting game is a simple game which has a weighted form, (W, q), where $W = (w_1, \ldots, w_n) \in (\mathbb{R}^+)^n$ corresponds to the weights of agents in I, and $q \in \mathbb{R}^+$ is the quota of the game. A coalition S wins if the total weight of S, $w(S) = \sum_{i \in S} w_i \geq q$, which implies that $v(S) = 1$. A WVG G of n agents with quota q is denoted by $G = [q; w_1, \ldots, w_n]$. Note also that $\frac{1}{2}w(I) < q \leq w(I)$.

Definition 3. *Critical Agent*

An agent $i \in S$ is critical to a winning coalition S if $w(S) \geq q$ and $w(S \backslash \{i\}) < q$.

Definition 4. *Banzhaf Power Index*

The Banzhaf power index computation for an agent i is the proportion of the number of coalitions i is critical compared to the total number of coalitions any agent in the game is critical. The Banzhaf index, $\beta_i(G)$, for each agent i in a WVG G is given by

$$\beta_i(G) = \frac{\eta_i(G)}{\sum_{j \in I} \eta_j(G)} \tag{1}$$

where $\eta_i(G)$ is the number of winning coalitions in which agent i is critical in game G.

Consider using the Banzhaf index to compute the power of the first agent in the WVG $G = [10, 7, 6, 4; 18]$. There are four winning coalitions in the game: $\{\{\underline{10}, \underline{7}, \underline{6}\}, \{\underline{10}, \underline{6}, \underline{4}\}, \{\underline{10}, \underline{7}, \underline{4}\}, \{\underline{10}, 7, 6, 4\}\}$. The critical agents in each of the winning coalitions are underlined. The first agent (weight 10) is critical four times. The second (weight 7), third (weight 6), and fourth (weight 4) agents are critical in only two winning coalitions each. So, the Banzhaf index of the first agent, $\beta_1(G) = \frac{4}{4+2+2+2} = 0.40$. Similarly, the power of each of the second, third, and fourth agent is $\frac{2}{4+2+2+2} = 0.20$.

Definition 5. *Shapley-Shubik Power Index*

The Shapley-Shubik index quantifies the marginal contribution of an agent to the grand coalition. Each permutation of the agents is considered. We term an agent *pivotal* in a permutation if the agents preceding it do not form a winning coalition, but by including this agent, a winning coalition is formed. We specify the computation of the index using notation of (Aziz et al., 2011). Denote by π, a permutation of the agents,

so $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$, and by Π the set of all possible permutations. Denote by $S_\pi(i)$ the predecessors of agent i in π, i.e., $S_\pi(i) = \{j : \pi(j) < \pi(i)\}$. The Shapley-Shubik index, $\varphi_i(G)$, for each agent i in a WVG G:

$$\varphi_i(G) = \frac{1}{n!}\sum_{\pi\in\Pi}[v(S_\pi(i)\cup\{i\}) - v(S_\pi(i))]. \qquad (2)$$

3.2 Formal Problem Definition

Let $G = [q; w_1,\ldots,w_n]$ be a WVG of n agents. Let $k \in \mathbb{N}$, $2 \le k < n$. Consider a manipulators' coalition S of k agents which is a k-subset of the n-set I. We assume that S contains k distinct elements chosen from I. Suppose the manipulators in S merge their weights to form a bloc denoted by $\&S$, i.e., agents $i \in S$ have been assimilated into the bloc $\&S$, then, we have a new set of agents in the game after merging. Thus, the initial game G of n agents has been altered by the manipulators to give a new game G' of $n-k+1$ agents consisting of the bloc, $\&S$, and other agents not in the bloc, i.e., $I\backslash S$.

Denote by $(\beta_1(G),\ldots,\beta_n(G)) \in [0,1]^n$ the Banzhaf power of agents in a WVG G of n agents. Thus, for the strategic agents $i \in S$ with power $\beta_i(G)$ in game G, the sum of the power of the k manipulators in S is $\sum_{i\in S}\beta_i(G)$, while that of the bloc formed by the manipulators in the altered game G' is $\beta_{\&S}(G')$. The ratio $\tau = \frac{\beta_{\&S}(G')}{\sum_{i\in S}\beta_i(G)}$ compares the power of the bloc in G' to the sum of the original power of the agents in the merged bloc. τ gives a factor of the power gained or lost when strategic agents $i \in S$ alter G to give G'. We say that β is susceptible to manipulation if there exists a game G' such that $\tau > 1$; the merging is termed *advantageous*. If $\tau < 1$, then the merging is *disadvantageous*, while the merging is *neutral* when $\tau = 1$.

3.3 Examples of Manipulation by Merging

We provide next illustration of manipulation by merging in WVGs using the Banzhaf power index to compute agents' power. The strategic agents in each game are shown in bold.

Example 1. *Advantageous Merge*

Let $G = [28; \mathbf{8}, 8, 8, 6, \mathbf{5}, 5, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{2}]$ be a WVG, i.e., a game with quota, $q = 28$, and ten agents, $1,2,\ldots,10$. The power of the strategic agents are, $\beta_1(G) \approx 0.1784$,$\beta_5(G) \approx 0.0843$,$\beta_7(G) = \beta_8(G) = \beta_9(G) = \beta_{10}(G) \approx 0.0412$. Their cumulative power is ≈ 0.4275. Suppose the manipulators form a bloc and alter G by merging their weights into a single weight as follows; $G' = [28; \mathbf{23}, 8, 8, 6, 5]$. The power of this bloc is $\beta_{\&S}(G') = \beta_1(G') = 0.7895 > 0.4275$. The factor by which the bloc gains is $\frac{0.7895}{0.4275} \approx 1.85$

Example 2. *Disadvantageous Merge*

Let $G = [56; 10, 9, 9, \mathbf{9}, 8, \mathbf{7}, 6, \mathbf{6}, \mathbf{2}, \mathbf{1}]$ be a WVG of ten agents. The power of the strategic agents are, $\beta_4(G) \approx 0.1238$, $\beta_6(G) = \beta_8(G) \approx 0.1139$, $\beta_9(G) \approx 0.0248$, and $\beta_{10}(G) \approx 0.0149$. Their cumulative power is ≈ 0.3913. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [56; \mathbf{25}, 10, 9, 9, 8, 6]$. The power of this bloc is $\beta_{\&S}(G') = \beta_1(G') \approx 0.2308 < 0.3913$. The factor by which the bloc loses is $\frac{0.2308}{0.3913} \approx 0.58$

Example 3. *Neutral Merge*

Let $G = [3; 2, \mathbf{1}, \mathbf{1}, 1]$ be a WVG of four agents. The power of the strategic agents are, $\beta_2(G) = \beta_3(G) \approx 0.1666667$. Their cumulative power is ≈ 0.3333334. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [3; 2, \mathbf{2}, 1]$. The power of this bloc is $\beta_{\&S}(G') = \beta_2(G') \approx 0.3333333$. Rounding the cumulative power of the manipulators (in G) and that of the bloc (in G') to 0.3333 shows that the strategic agents neither gain nor lose power by merging their weights in this case.

We have shown that strategic agents may gain power, lose power, or their power may remain the same when they engage in manipulation by merging.

4 BANZHAF INDEX BOUNDS

Consider a WVG G with quota q involving agents I. Let w_i be the weight of an agent $i \in I$. The games we consider are those for which $w_i < q$, and when a manipulators' bloc $S \subset I$ merges in an altered game G' to form a bloc $\&S$, the weight of the bloc, $w(\&S) < q$. We present in this section upper and lower bounds that are within constant factors to characterize the effects of merging when there are $k = 2$ strategic agents in a WVG using the Banzhaf power index.

4.1 Upper Bound

Theorem 1. *Let $G = [q; w_1,\ldots,w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Banzhaf power, $\beta_{\&S}(G')$, of the bloc in the new game, $\beta_{\&S}(G') \le 3(\beta_{m_1}(G) + \beta_{m_2}(G))$.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2 (with weights w_1 and w_2, respectively), from the original game G that would like

to merge their weights, and form a bloc $\&S$ in an altered game G'. Assume without loss of generality that $w_1 \leq w_2$. Recall that $\eta_i(G)$ is the number of winning coalitions for which an agent i is critical in WVG G.

We first bound the number of winning coalitions in G for which the manipulators are critical. Let Γ_G be the set of all possible coalitions of the n agents in G. Also, let Γ_{G-2} be the set of all possible coalitions of the remaining $n-2$ non-manipulating agents in G, i.e., not including agents m_1 and m_2. Consider any coalition $c \in \Gamma_{G-2}$ such that $w(c) < q$. Suppose we add strategic agents m_1 and/or m_2 to c and have a resulting winning coalition $c^* \in \Gamma_G$. Let Γ_G^* be the set of all possible coalitions c^* such that at least one of m_1 or m_2 is critical in G. We define the subsets $C_i \subseteq \Gamma_G^*$ as follows:

$C_1 = \{C \subseteq \Gamma_{G-2} : w(C) < q, w(C) + w_1 \geq q\}$
$C_2 = \{C \subseteq \Gamma_{G-2} : w(C) < q, w(C) + w_2 \geq q\}$
$C_3 = \{C \subseteq \Gamma_{G-2} : w(C) + w_1 < q, w(C) + w_1 + w_2 \geq q\}$
$C_4 = C_5 = \{C \subseteq \Gamma_{G-2} : w(C) + w_1 < q, w(C) + w_2 < q, w(C) + w_1 + w_2 \geq q\}$.

Thus,

$$\eta_{m_1}(G) + \eta_{m_2}(G) = \sum_{i=1}^{5} |C_i|. \tag{3}$$

Note that C_1 are winning coalitions in G for which m_1 is critical, and which does not include m_2. C_2 are winning coalitions in G for which m_2 is critical, and which does not include m_1. C_3 are winning coalitions in G for which m_2 is critical, and which includes m_1. Finally, C_4 and C_5 are winning coalitions in G for which both m_1 and m_2 are critical. We set $C_4 = C_5$ since we need to count a winning coalition twice when both m_1 and m_2 are critical in that coalition. So, $|C_4|$ counts the winning coalitions for m_1 while $|C_5|$ counts the winning coalitions for m_2.

Now, we bound the number of coalitions for which the non-manipulators are critical in game G. Note that there are two possibilities for an arbitrary non-manipulating agent $j \in I \setminus \{m_1, m_2\}$ to be critical in a winning coalition in game G:

$S_1 = \{S \subseteq \Gamma_{G-2} \setminus \{j\} : w(S) < q, w(S) + w_j \geq q\}$
$S_2 = \{S \subseteq I \setminus \{j\} : m_1 \in S \vee m_2 \in S, w(S) < q, w(S) + w_j \geq q\}$[1].

S_1 are the winning coalitions in G for which agent j is critical, and which does not include both m_1 and m_2. S_2 are the winning coalitions in G for which agent j is critical, and which include at least one of m_1 or m_2. Thus, we have

$$\eta_j(G) = |S_1| + |S_2|. \tag{4}$$

Furthermore, we bound the number of coalitions in the altered game G' for which a manipulators' bloc, $\&S$, is critical. Let $c \in \Gamma_{G-2}$ be a coalition in game G such that $w(c) < q$. We define a function f such that a coalition, $f(c) = c \cup \{\&S\}$, of agents in game G' is winning if and only if at least one of the coalitions, $c \cup \{m_1\}$, $c \cup \{m_2\}$, or $c \cup \{m_1, m_2\}$, is winning in G.

We claim that for any c such that $c \cup \{m_1\} \in C_1$, it must also be the case that $c \cup \{m_2\} \in C_2$, since $w_1 \leq w_2$. Observe that the *two* winning coalitions, $c \cup \{m_1\} \in C_1$ and $c \cup \{m_2\} \in C_2$, of the manipulators, m_1 and m_2, from G, correspond to *exactly one* winning coalition $f(c)$ of the bloc in G'. Similarly, for the case when $c \cup \{m_1\} \notin C_1$ and $c \cup \{m_2\} \in C_2$, it must also be true that, $c \cup \{m_1, m_2\} \in C_3$. This is because since if m_1 is not critical in $c \cup \{m_1\}$, it also cannot be critical in the winning coalition $c \cup \{m_1, m_2\} \in C_3$, where m_2 is present. Note, however, that, only m_2 is critical for coalition $c \cup \{m_1, m_2\} \in C_3$ by definition. Thus, again, the *two* winning coalitions, $c \cup \{m_2\} \in C_2$ and $c \cup \{m_1, m_2\} \in C_3$, of m_2, from G correspond to *exactly one* winning coalition $f(c)$ of the bloc in G'. Finally, both m_1 and m_2 are critical for any coalition $c \cup \{m_1, m_2\} \in C_4$ and $c \cup \{m_1, m_2\} \in C_5$ by definition. The *two* winning coalitions, $c \cup \{m_1, m_2\} \in C_4$ and $c \cup \{m_1, m_2\} \in C_5$, of m_1, m_2, from G correspond to *exactly one* winning coalition $f(c)$ of the bloc in G'. We conclude that the number of coalitions for which a manipulators' bloc, $\&S$, is critical in G' is one-half of the number of times the manipulators, m_1 and m_2, are critical in G. Thus,

$$\eta_{\&S}(G') = \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2}. \tag{5}$$

It remains to bound the number of coalitions in G' for which the non-manipulators are critical. Note that because the power of the manipulators' bloc is the ratio of the number of winning coalitions in which the bloc is involved divided by the total number of winning coalitions involving all agents, the most power will be obtained by the bloc when the manipulators' bloc is involved in highest number of winning coalitions and the non-manipulators are involved in the least number of winning coalitions. Let $S \in S_1$. Clearly, since $m_1 \notin S$ and $m_2 \notin S$, S remains unchanged from G to G'. Hence, for this case, the non-manipulating agent j remains critical in G' for $|S_1|$ number of winning coalitions. Similarly, let $S \in S_2$. Since at least one of m_1 or m_2 is in S, the three possible coalitions for j to be critical for S in G are:

[1] $a \vee b$ is the *or* of a and b. At least one of the two simple parts of the compound proposition is required to be true.

$S \cup \{m_1, j\}$, $S \cup \{m_2, j\}$, and $S \cup \{m_1, m_2, j\}$. However, these *three* coalitions correspond to *exactly one* winning coalition, $S \cup \{\&S, j\}$, of the bloc in G'. Thus,

$$\begin{aligned}
\eta_j(G') &= |S_1| + \frac{|S_2|}{3} && (6)\\
&= \frac{3|S_1| + |S_2|}{3} && (7)\\
&= \frac{|S_1| + |S_2| + 2|S_1|}{3} && (8)\\
&= \frac{\eta_j(G)}{3} + \frac{2|S_1|}{3} && (9)\\
&\geq \frac{\eta_j(G)}{3}. && (10)
\end{aligned}$$

We compute the Banzhaf power index of the bloc $\&S$ in game G' using (5) and (10):

$$\begin{aligned}
\beta_{\&S}(G') &= \frac{\eta_{\&S}(G')}{\eta_{\&S}(G') + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&= \frac{\frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2}}{\frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2} + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&= \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{\eta_{m_1}(G) + \eta_{m_2}(G) + 2\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&\leq \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{\eta_{m_1}(G) + \eta_{m_2}(G) + \frac{2}{3}\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\leq \frac{3(\eta_{m_1}(G) + \eta_{m_2}(G))}{3(\eta_{m_1}(G) + \eta_{m_2}(G)) + 2\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\leq \frac{3(\eta_{m_1}(G) + \eta_{m_2}(G))}{\eta_{m_1}(G) + \eta_{m_2}(G) + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\leq 3(\beta_{m_1}(G) + \beta_{m_2}(G))
\end{aligned}$$

□

4.2 Lower Bound

Theorem 2. *Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Banzhaf power, $\beta_{\&S}(G')$, of the bloc in the new game, $\beta_{\&S}(G') \geq \frac{\beta_{m_1}(G) + \beta_{m_2}(G)}{2}$.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let S_1 and S_2 be as defined in Theorem 1. We are interested in finding the minimum factor of power that can be gained by a merged bloc $\&S$ in game G'. Note again that, because the power of the bloc is the ratio of the number of winning coalitions in which the bloc is involved in divided by the total number of winning coalitions involving all agents, the least power will be obtained by the bloc when the manipulators' bloc is involved in the fewest number of winning coalitions and the non-manipulating agents are involved in the highest number of winning coalitions. Thus, we seek to find the maximum number of winning coalitions in which the non-manipulating agents can participate. The case which yields the maximum number of winning coalitions in game G is that agent j is critical for the three coalitions, $S \cup \{m_1, j\}$, $S \cup \{m_2, j\}$, and $S \cup \{m_1, m_2, j\}$ involving the manipulators. This is so because coalitions, S_1, contributing to the overall total remains the same from G to G'. Thus, as before:

$$\begin{aligned}
\eta_j(G') &= |S_1| + \frac{|S_2|}{3} && (11)\\
&\leq |S_1| + \frac{|S_2|}{3} + \frac{2|S_2|}{3} && (12)\\
&\leq |S_1| + |S_2| && (13)\\
&\leq \eta_j(G). && (14)
\end{aligned}$$

We compute the Banzhaf power index of the bloc $\&S$ in game G' using (5) and (14):

$$\begin{aligned}
\beta_{\&S}(G') &= \frac{\eta_{\&S}(G')}{\eta_{\&S}(G') + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&= \frac{\frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2}}{\frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2} + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&= \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{\eta_{m_1}(G) + \eta_{m_2}(G) + 2\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}\\
&\geq \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{\eta_{m_1}(G) + \eta_{m_2}(G) + 2\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\geq \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2(\eta_{m_1}(G) + \eta_{m_2}(G)) + 2\sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\geq \frac{1}{2} \cdot \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{\eta_{m_1}(G) + \eta_{m_2}(G) + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G)}\\
&\geq \frac{\beta_{m_1}(G) + \beta_{m_2}(G)}{2}.
\end{aligned}$$

□

5 CONCLUSIONS

This paper investigates the effects of manipulation by merging in weighted voting games. Manipulation by merging refers to a dishonest behavior where two or more strategic agents merge their weights to form a single bloc in anticipation of power increase. We consider a prominent payoff concept, the Banzhaf index, that is used in evaluating agents' power in such games. Our focus is on the characterization of the extent to which agents may gain in such manipulation.

The concern of this research is based on the assumption that the inability to limit (or understand) the

Table 2: Summary of bounds for manipulation by merging in weighted voting games.

Bounds	# Strategic agents	Shapely-Shubik Index	Banzhaf Index
Upper	$k = 2$	Lasisi & Lasisi '15	This paper
	$k > 2$	?	?
Lower	$k = 2$	Lasisi & Lasisi '15	This paper
	$k > 2$	?	?

effects of this manipulation may undermine the confidence agents have in decisions made via weighted voting games. If the results from this class of games are not seen as fair, agents may refuse to abide by decisions made in this manner. Thus, specifically, we propose two new and non-trivial bounds for this problem when there are $k = 2$ strategic agents in the games. The two bounds are also found to be within constant factors. Our results complement those of a previous work, (Lasisi and Lasisi, 2015), which propose two tight bounds (upper and lower) for this problem, also for the case when the number k of strategic agents is 2, but using the Shapley-Shubik index. Table 2 provides a summary on the state of the arts on the bounds for manipulation by merging in weighted voting games using both the Shapley-Shubik and Banzhaf power indices.

There are several areas of ongoing research on this problem. Here are some directions for future work. We have considered the case when the number, k, of strategic agents in a weighted voting game is 2. As shown in Examples 1 and 2, it is also possible for the number of strategic agents to be more than 2. Thus, it will be interesting to see non-trivial upper and lower bounds for this problem for the case, $k > 2$, using both the Shapley-Shubik and Banzhaf indices to compute agents' power. Furthermore, our immediate future work is to complement these theoretical results with empirical evaluations to see the extent of the factors for beneficial merges to strategic agents in practice. Finally, developing methods to reduce the effects of manipulation by merging in weighted voting games is an interesting research problem to consider.

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