ANNEXATIONS AND MERGING IN WEIGHTED VOTING GAMES
The Extent of Susceptibility of Power Indices

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Abstract: This paper discusses weighted voting games and two methods of manipulating those games, called *annexation* and *merging*. These manipulations allow either an agent, called an *annexer* to take over the voting weights of some other agents in the game, or the coming together of some agents to form a *bloc* of manipulators to have more power over the outcomes of the games. We evaluate the extent of susceptibility to these manipulations in weighted voting games of the following prominent power indices: Shapley-Shubik, Banzhaf, and Deegan-Packel indices. We found that for *unanimity* weighted voting games of *n* agents and for the three indices: the *manipulability*, (i.e., the extent of susceptibility to manipulation) via annexation of any one index does not dominate that of other indices, and the upper bound on the extent to which an annexer may gain while annexing other agents is at most *n* times the power of the agent in the original game. Experiments on *non unanimity* weighted voting games suggest that the three indices are highly susceptible to manipulation via annexation while they are less susceptible to manipulation via merging. In both annexation and merging, the Shapley-Shubik index is the most susceptible to manipulation among the indices.

1 INTRODUCTION

Weighted voting games (WVGs) are mathematical abstractions of voting systems. In a voting system, voters express their opinions through their votes by electing candidates to represent them or influence the passage of bills. Each member of the set of voters, V, has an associated weight $w : V \to Q^+$. A voter's weight is the number of votes controlled by the voter, and this is the maximum number of votes she is permitted to cast. The *homogeneous voting system* is a special case in which all voters have unit weight (Levchenkova and Levchenkov, 2002). In our context, a subset of agents, called the *coalition*, wins in a WVG, if the sum of the weights of the individual agents in the coalition meets or exceeds a certain threshold called the *quota*. In the more traditional homogeneous voting system, the winning coalition is determined by the majority of the agents. However, in WVGs with all agents having different weights, a coalition with sum of the individual agents' weights meeting or exceeding the quota determines the winning coalition.

It is natural to naively think that the numerical weight of an agent directly determines the corresponding strength of the agent in a WVG. The measure of the strength of an agent is termed its *power*. This is the ability of an agent to influence the decision-making process. Consider, for example, a WVG of three voters, a_1, a_2, and a_3 with respective weights $6, 3,$ and 1. When the quota for the game is 10, then a coalition consisting of all the three voters is needed to win the game. Thus, each of the voters are of equal importance in achieving the winning coalition. Hence, they each have equal power irrespective of their weight distribution, in that every voter is necessary for a win. The three prominent power indices for measuring agents' power are the *Shapley-Shubik*, *Banzhaf*, and *Deegan-Packel* indices (Matsui and Matsui, 2000). All the three methods assign equal power to the voters in this example.

This paper discusses WVGs and two methods of manipulating those games, called *annexation* and *merging* (Aziz and Paterson, 2009). In annexation, a strategic agent, termed an *annexer*, may alter a game by taking over the voting weights of some other agents in the game in order to use the weights in her favor. As a straightforward example of annexation, consider when a shareholder buys up the voting shares of some other shareholders (Machover and Felsenthal, 2002). We refer to the agents whose voting shares were bought as the *assimilated voters*. The new game consists of the previous agents in the original game whose weights were not annexed by the strategic agent and the *bloc* of agent made up of the annexer and the assimilated voters. The annexer also incurs some *annexation cost* to allow purchasing the votes of the assimilated voters. In this situation, only the annexer benefits from the annexation as the power of the bloc in the new game is compared to the power of the annexer in the original game.

On the other hand, merging is the voluntary coordinated action of would-be manipulators who come together to form a bloc. The agents in the bloc are also assumed to be assimilated voters since they can no more vote as individual voters in the new game, rather as a bloc. The new game consists of the previous agents in the original game that are not assimilated as well as the bloc formed by the assimilated voters. The power of the bloc in the new game is compared to the sum of the individual powers of all members of the assimilated bloc in the original game. No annexation costs occur as individual voters in the bloc are compensated via power. All the agents in the bloc benefit from the merging in case of power increase, having agreed on how to distribute the gains from their collusion.

In both annexation and merging, strategic agents anticipate that the value of their power in the new games to be at least the value of their power in the original games. (Machover and Felsenthal, 2002) show that this anticipation of power increase due to annexation is always achieved by the annexer when the Shapley-Shubik index is used to compute power. This is not true for the Banzhaf index as the power of the annexer can decrease compared to its power in the original game. For the case of merging and for both the Shapley-Shubik and Banzhaf indices, there are situations where the power of the bloc decreases compared to the sum of the individual powers of all members of the assimilated bloc in the original games.

To date, the more detailed analysis of players merging into blocs remains unexplored (Aziz and Paterson, 2009). This paper evaluates the susceptibility to manipulation via annexation and merging in WVGs of the following power indices: Shapley-Shubik, Banzhaf, and Deegan-Packel indices. This is the extent to which strategic agents may gain power with respect to the original games they manipulate. We provide empirical analysis of susceptibility to annexation and merging in WVGs among the three indices. The main results of this paper are the following:

1. For any *unanimity* WVGs of n agents:

   a. Contrary to (Aziz and Paterson, 2009) that for both Shapley-Shubik and Banzhaf indices it is advantageous for a player to annex, we show that this is not true in its entirety. Apart from the fact that annexation always increases the power of other agents that are not annexed by the same *factor of increment* as the annexer achieved, the annexer also incurs *annexation costs* that reduce the benefit the agent thought it gained.

   b. Using the Shapley-Shubik, Banzhaf, and Deegan-Packel indices to compute power, the *manipulability* of any one index does not dominate the manipulability of other indices.

   c. The upper bound on the extent to which a strategic agent may gain (i.e., the factor of increment) while annexing other agents in the altered game is at most n times the power of the agent in the original game. The result holds for the Shapley-Shubik, Banzhaf, and the Deegan-Packel power indices.

2. The Shapley-Shubik, Banzhaf, and the Deegan-Packel indices are all *highly* susceptible to manipulation via annexation in *non unanimity* WVGs. However, the Shapley-Shubik index is the most susceptible of the three indices.

3. Unlike manipulation via annexation in the non unanimity WVGs, the Shapley-Shubik, Banzhaf, and the Deegan-Packel indices are all *less* susceptible to manipulation via merging. Again, the Shapley-Shubik index is the most susceptible of the three indices.

4. Finally, the Shapley-Shubik index manipulability dominates that of the Banzhaf index, which in turn dominates that of the Deegan-Packel index for both manipulation via annexation and merging in non unanimity WVGs.

The remainder of the paper is organized as follows. Section 2 discusses related work. Section 3 provides the definitions and notations used in the paper. In Section 4, we provide examples using the three power indices to illustrate manipulation via annexation and merging in WVGs. Section 5 considers unanimity and non unanimity WVGs. We also provide evaluation of manipulation via annexation for unanimity WVGs. Section 6 provides empirical evaluation of susceptibility of the three power indices to manipulations via annexation and merging for non unanimity WVGs. We conclude in Section 7.

2 RELATED WORK

Weighted voting games and power indices are widely studied (Matsui and Matsui, 2000; Leech, 2002; Alonso-Meijide and Bowles, 2005; Bachrach et al., 2008; Aziz and Paterson, 2009). WVGs have many applications, including economics, political science, neuroscience, threshold logic, reliability theory, distributed systems (Aziz et al., 2007), and multiagent systems (Bachrach and Elkind, 2008). Prominent real-life situations where WVGs have found applications include the United Nations Security Council, the Electoral College of the United States and the International Monetary Fund (Leech, 2002; Alonso-Meijide and Bowles, 2005).

The study of WVGs has also necessitated the need to fairly determine the power of players in a game. This is because the power of a player in a game provides information about the relative importance of that player when compared to other players. To evaluate players' power, prominent power indices such as Shapley-Shubik, Banzhaf, and Deegan-Packel indices are commonly employed (Matsui and Matsui, 2000). These indices satisfy the axioms that characterize a power index, have gained wide usage in political arena, and are the main power indices found in the literature (Laruelle, 1999). These power indices have been defined on the framework of subsets of winning coalitions in the game they seek to evaluate. A wide variation in the results they provide can be observed. This is due to the different definitions and methods of computation of the associated subsets of the winning coalitions. Then, comes the question of which of the power indices is the most resistant to manipulation in a WVG. The choice of a power index depends on a number of factors, namely, the a priori properties of the index, the axioms characterizing the index, and the context of decision making process under consideration (Laruelle, 1999).

The three indices we consider measure the influence of voters differently. There are many situations where their values are the same for similar games. However, there exists an important example of the US federal system while using the Shapley-Shubik and Banzhaf indices where they do not agree (Kirsch and Langner, 2010). According to (Laruelle and Valenciano, 2005), and (Kirsch, 2007), the decision of which index to use in evaluating a voting situation is largely dependent on the assumptions about the voting behavior of the voters. When the voters are assumed to vote completely independently of each other, the Banzhaf index has been found to be appropriate. On the other hand, Shapley-Shubik index should be employed when all voters are influenced by a common belief on their choices. Deegan-Packel index is appealing in that it assigns powers based on size of the winning coalition, thus giving preference to smaller coalitions (which may be easier to form).

Under certain assumptions in the WVGs, computing the power indices of voters using any of Shapley-Shubik, Banzhaf, or Deegan-Packel indices is NP-hard (Matsui and Matsui, 2000). (Deng and Papadimitriou, 1994) also show that computing the Shapley value in WVGs is #P-complete. However, the power of voters using any of the three indices can be computed in pseudo-polynomial time by dynamic programming (Garey and Johnson, 1979; Matsui and Matsui, 2000). There are also approximation algorithms for computing the Shapley-Shubik and Banzhaf power indices (Bachrach et al., 2008).

(Bachrach and Elkind, 2008) have studied a form of manipulation in WVGs called *false name manipulation*. In false name manipulation, a strategic agent may alter a game in anticipation of power increase by splitting its weight among several false identities that are not in the original game. They use the Shapley-Shubik index to evaluate agents' power and consider the case when an agent splits into exactly two false identities. The extent to which agents increase or decrease their Shapley power is also bounded. Similar results using Banzhaf index were obtained by (Aziz and Paterson, 2009). Furthermore, (Lasisi and Allan, 2010) extends existing work by (Bachrach and Elkind, 2008), and (Aziz and Paterson, 2009). Their work empirically considers the effects of false name manipulation in WVGs when an agent splits into more than two identities. Results of their experiments suggest that the three indices are susceptible to false name manipulation in WVGs. However, that the Deegan-Packel index is more susceptible than the Shapley-Shubik and Banzhaf indices.

As mentioned in the introduction, very little work exists on manipulation via annexation and merging in WVGs, and the more detailed analysis of players merging into blocs, until now, has remained unexplored (Aziz and Paterson, 2009). We discuss some notable exceptions. (Machover and Felsenthal, 2002), prove that if a player annexes other players, then the annexation is always advantageous for the annexer using the Shapley-Shubik index. The annexation can be advantageous or disadvantageous using the Banzhaf index. For the case of merging, and for both the Shapley-Shubik and Banzhaf indices, merging can be advantageous or disadvantageous. (Aziz and Paterson, 2009) show that for some classes of WVGs, and for both Shapley-Shubik and Banzhaf indices, it is disadvantageous for a coalition to merge, while advantageous for a player to annex. They also prove some NP-hardness results for annexation and merging in WVGs. They show that for both Shapley-Shubik and Banzhaf indices, finding a beneficial annexation is NP-hard. Also, determining if there exists a beneficial merge is NP-hard for the Shapley-Shubik index.

(Machover and Felsenthal, 2002), and (Aziz and Paterson, 2009) have shown that it can be advantageous for strategic agents to engage in annexation or merging for Shapley-Shubik and Banzhaf indices in some classes of WVGs. The authors stop short of addressing the question of upper bounds on the extent to which strategic agents may gain with respect to the games they manipulate. In view of this, our work differ from those of these authors. We extend the work of (Lasisi and Allan, 2010), as we study the susceptibility of the three indices to manipulation via annexation and merging. We empirically consider the extent to which strategic agents may gain by engaging in such manipulation and show how the susceptibility among the indices compares for different WVGs.

3 DEFINITIONS & NOTATIONS

We give the following definitions and notations used throughout the paper.

Weighted Voting Game. Let I = $\{1,\cdots,n\}$ be a set of n agents. Let $\mathbf{w} = \{w_1,\cdots,w_n\}$ be the corresponding positive integer weights of the agents. Let S be a non empty set of agents. $S \subseteq I$ is a *coalition*. A WVG G with *quota* q involving agents I is defined as $G = [w_1,\cdots,w_n;q]$. Denote by $w(S)$, the weight of a coalition S derived from the summation of the individual weights of agents in S i.e., $w(S) = \sum_{i\in S} w_i$. A coalition, S, wins in the game G if $w(S) \geq q$ otherwise it loses. q is constrained as follows $\frac{1}{2}w(I) < q \leq w(I)$. Thus, disjoint winning coalitions cannot emerge.

Simple Voting Game. Each of the coalitions $S \subseteq I$ has an associated value function $v : S \to \{0,1\}$. The value 1 implies a win for the coalition and 0 a loss. In the game $G, v(S) = 1$ if $w(S) \geq q$ and 0 otherwise.

Dummy and Critical Agents. An agent $i \in S$ is *dummy* if its weight in S is not needed for S to be a winning coalition, i.e., $w(S\backslash\{i\}) \geq q$. Otherwise, it is *critical* to S, i.e., $w(S) \geq q$ and $w(S\backslash\{i\}) < q$.

Unanimity Weighted Voting Game. A WVG in which there is a single winning coalition and every agent is critical to the coalition is *unanimity weighted voting game*.

Shapley-Shubik Power Index. The Shapley-Shubik power index is one of the oldest power indices and has been used widely to analyze political power. The index quantifies the marginal contribution of an agent to the grand coalition. Each agent in a permutation is given credit for the win if the agents preceding it do not form a winning coalition but by adding the agent in question, a winning coalition is formed. The power index is dependent on the number of permutations for which an agent is critical. For the $n!$ permutations of agents used in determining the Shapley-Shubik index, there exists exactly one critical agent in each of the permutations. Denote by Π the set of all permutations of n agents in a WVG G. Let $\pi \in \Pi$ define a one-to-one mapping where $\pi(i)$ is the position of the ith agent in the permutation order. Denote by $S_\pi(i)$, the predecessors of agent i in π, i.e., $S_\pi(i) = \{j : \pi(j) < \pi(i)\}$. The Shapley-Shubik index, $\varphi_i(G)$, of agent i in G is given by

$$\varphi_i(G) = \frac{1}{n!}\sum_{\pi\in\Pi}[v(S_\pi(i)\cup\{i\}) - v(S_\pi(i))] \qquad (1)$$

Banzhaf Power Index. Another index that has also gained wide usage in the political arena is the Banzhaf power index. Unlike the Shapley-Shubik index, its computation depends on the number of winning coalitions in which an agent is critical. There can be more than one critical agent in a particular winning coalition. The Banzhaf index, $\beta_i(G)$, of agent i in the same game, G, as above is given by

$$\beta_i(G) = \frac{\eta_i(G)}{\sum_{i\in I}\eta_i(G)} \qquad (2)$$

where $\eta_i(G)$ is the number of coalitions in which i is critical in G.

Deegan-Packel Power Index. The Deegan-Packel power index is also found in the literature for computing power indices. The computation of this power index for an agent i takes into account both the number of all the minimal winning coalitions (MWCs) in the game as well as the sizes of the MWCs having i as a member (Matsui and Matsui, 2000). Thus, it is more impressive to be one in three (who elicited the win) rather than one in ten. A winning coalition $C \subseteq I$ is a MWC if every proper subset of C is a losing coalition, i.e., $w(C) \geq q$ and $\forall T \subset C, w(T) < q$. The Deegan-Packel power index, $\gamma_i(G)$, of an agent i in G is given by

$$\gamma_i(G) = \frac{1}{|MWC|}\sum_{S\in MWC_i}\frac{1}{|S|} \qquad (3)$$

where MWC_i are the sets of all MWCs in G that include i.

Susceptibility of Power Index to Manipulation. Consider a coalition $S \subset I$, let $\&S$ defines a bloc of assimilated voters formed by agents in S.

Annexation : Let Φ be a power index. Denote by $\Phi_i(G)$, the power of an agent i in a WVG G. Suppose i alters G by annexing a coalition S. Let G' be the resulting game after the annexation. We say that Φ is susceptible to manipulation via annexation if there exists a G', such that $\Phi_{\&(S\cup\{i\})}(G') > \Phi_i(G)$; the annexation is termed *advantageous*. If $\Phi_{\&(S\cup\{i\})}(G') < \Phi_i(G)$, then the annexation is *disadvantageous*.

Merging : Let Φ be a power index. Denote by $\Phi_i(G)$, the power of an agent i in a WVG G. Suppose a coalition, S, alters G by merging into a bloc. Let G' be the resulting game after the merging. We say that Φ is susceptible to manipulation via merging if there exists a G', such that $\Phi_{\&S}(G') > \sum_{i\in S}\Phi_i(G)$; the merging is termed *advantageous*. If $\Phi_{\&S}(G') < \sum_{i\in S}\Phi_i(G)$, then the merging is *disadvantageous*.

Factor of Increment (Decrement). Let Φ be a power index. Denote by $\Phi_i(G)$, the power of an agent i in a WVG G. Let G' be the resulting game when i alters G by manipulation. The factor of increment (resp. decrement) of the original power from the manipulation is $\frac{\Phi_i(G')}{\Phi_i(G)}$. The value represents an increment (or gain) if it is greater than 1 and decrement (or loss) if it is less than 1. The factor of increment provides an indication of the extent of susceptibility of power indices to manipulation. A higher factor of increment by a power index in a game indicates that the index is more susceptible to manipulation in that game.

Domination of Manipulability. Let Φ and Θ be two different power indices. Denote by $\Phi_i(G)$ and $\Theta_i(G)$, the respective power of an agent i in a WVG G as determined by Φ and Θ. Let i be an annexer. Suppose the corresponding power of the agent in a new game G' when i alters G by assimilating agents S are $\Phi_{\&(S\cup\{i\})}(G')$ and $\Theta_{\&(S\cup\{i\})}(G')$. We say that the manipulability of one index say $\Phi^G_{G'}$, dominates the manipulability of another index $\Theta^G_{G'}$ for a particular game G, if the factor by which i gain in Φ is greater than the factor by which it gain in Θ, i.e., $\frac{\Phi_{\&(S\cup\{i\})}(G')}{\Phi_i(G)} > \frac{\Theta_{\&(S\cup\{i\})}(G')}{\Theta_i(G)}$, and hence, Φ is more susceptible to manipulation than Θ in G. The domination of manipulability can be similarly define for manipulation via merging.

4 ANNEXATIONS & MERGING

This section provides examples illustrating manipulation via annexation and merging in WVGs. The power of the strategic agents, i.e., the annexer or the bloc of manipulators, and the factor of increment (decrement) are also summarized in a table for each example using the three power indices.

4.1 Manipulation via Annexation

Example 1. *Annexation Advantageous. Let* $G = [\mathbf{5}, 8, 3, 3, \mathbf{4}, 2, 4; 18]$ *be a WVG. The assimilated agents are shown in bold, with agent* 1 *being the annexer. In the original game, the Deegan-Packel index of the annexer is* $\gamma_1(G) = 0.1722$*. In the new game,* $G' = [\mathbf{9}, 8, 3, 3, 2, 4; 18]$*, its Deegan-Packel index is* $\gamma_1(G') = 0.2604$*, a factor of increase of* 1.51.

Table 1: The annexer power in the original game $G = [\mathbf{5}, 8, 3, 3, \mathbf{4}, 2, 4; 18]$, the altered game $G' = [\mathbf{9}, 8, 3, 3, 2, 4; 18]$, and the factor of increment for the three indices.

Power Index	G	G'	Factor
Shapley-Shubik	0.1714	0.3500	2.04
Banzhaf	0.1712	0.3400	1.99
Deegan-Packel	0.1722	0.2604	1.51

Example 2. *Annexation Disadvantageous. Let* $G = [\mathbf{8}, 9, 9, 5, 7, \mathbf{3}, 9; 29]$ *be a WVG. The assimilated agents are shown in bold, with agent* 1 *being the annexer. In the original game, the Deegan-Packel index of the annexer is* $\gamma_1(G) = 0.1711$*. In the new game,* $G' = [\mathbf{11}, 9, 9, 5, 7, 9; 29]$*, its Deegan-Packel index is* $\gamma_1(G') = 0.1591$*, a factor of decrease of* 0.93.

Table 2: The annexer power in the original game $G = [\mathbf{8}, 9, 9, 5, 7, \mathbf{3}, 9; 29]$, the altered game $G' = [\mathbf{11}, 9, 9, 5, 7, 9; 29]$, and the factor of increment (decrement) for the three indices.

Power Index	G	G'	Factor
Shapley-Shubik	0.1786	0.2167	1.21
Banzhaf	0.1774	0.2167	1.22
Deegan-Packel	0.1711	0.1591	0.93

4.2 Manipulation via Merging

Example 3. *Merging Advantageous. Let* $G = [4, \mathbf{2}, 1, 1, 8, \mathbf{7}, \mathbf{4}; 17]$ *be a WVG. The assimilated agents are shown in bold. In the original game, the Deegan-Packel indices of these agents are,* $\gamma_2(G) = 0.0926, \gamma_6(G) = 0.1889$*, and* $\gamma_7(G) = 0.1704$*. Their cummulative power is* 0.4519*. In the new game,* $G' = [\mathbf{13}, 4, 1, 1, 8; 17]$*, the Deegan-Packel index of the bloc is* $\gamma_1(G') = 0.5000$*, a factor of increase of* 1.11.

Example 4. *Merging Disadvantageous. Let* $G = [5, \mathbf{8}, 3, 4, \mathbf{9}, 1, \mathbf{5}; 30]$ *be a WVG. The assimilated agents are shown in bold. In the original game, the Deegan-Packel indices of these agents are,* $\gamma_2(G) =$

Table 3: The cummulaive power of the assimilated agents in the original game $G = [4, \mathbf{2}, 1, 1, 8, \mathbf{7}, \mathbf{4}; 17]$, the power of the bloc in the altered game $G' = [\mathbf{13}, 4, 1, 1, 8; 17]$, and the factor of increment for the three indices.

Power Index	G	G'	Factor
Shapley-Shubik	0.4881	0.6667	1.37
Banzhaf	0.4851	0.6000	1.24
Deegan-Packel	0.4519	0.5000	1.11

$0.1833, \gamma_5(G) = 0.1333$, *and* $\gamma_7(G) = 0.1417$. *Their cummulative power is* 0.5083. *In the new game,* $G' = [\boldsymbol{22}, 5, 3, 4, 1; 30]$, *the Deegan-Packel index of the bloc is* $\gamma_1(G') = 0.3056$, *a factor of decrease of* 0.60.

Table 4: The cummulaive power of the strategic agents in the original game $G = [5, \mathbf{8}, 3, 4, \mathbf{9}, 1, \mathbf{5}; 30]$, the power of the bloc in the altered game $G' = [\mathbf{22}, 5, 3, 4, 1; 30]$, and the factor of decrement for the three indices.

Power Index	G	G'	Factor
Shapley-Shubik	0.6762	0.4667	0.69
Banzhaf	0.5789	0.3684	0.64
Deegan-Packel	0.5083	0.3056	0.60

5 WEIGHTED VOTING GAMES

This section considers manipulatons via annexation and merging for both unanimity and non unanimity WVGs. For the sake of simplicity in our discussion, we assume that for the case of manipulation via annexation, the annexer has enough resources to cover the annexation costs for all the agents it annexes. Also, we assume that only one of the agents is engaging in the annexation at a time. However, we are not oblivious of the fact that other agents also have similar motivations to engage in annexation in anticipation of power increase. For the case of manipulation via merging, we assume that the assimilated agents in the bloc can easily distribute the gains from their collusion among themselves in a fair and stable way. Thus, making them agree to engage in the manipulation if it is profitable.

5.1 Unanimity Weighted Voting Games

We recall that a WVG in which there is a single winning coalition and every agent is critical to the coalition is *unanimity* WVG. (Aziz and Paterson, 2009) have shown that for unanimity WVGs and for both the Shapley-Shubik and Banzhaf indices: *it is disadvantageous for a coalition to merge and advantageous for a player to annex other players*. We observe that these results naturally extend to the Deegan-Packel index too. This is because for unanimity WVGs, the definitions of the Shapley-Shubik, Banzhaf, and the Deegan-Packel indices using Formulas 1, 2, and 3, respectively, are equivalent. In fact, the power of all agents in any unanimity WVGs is the same for the three indices. In view of the annexation result of (Aziz and Paterson, 2009) above, and the fact that strategic agents are interested in annexations and merging that improve their power, we consider only manipulation via annexation for the unanimity WVGs.

Note that (Aziz and Paterson, 2009) have not considered the bounds on the extent to which strategic agents may gain with respect to games they manipulate. This is important as it provides motivations for strategic agents to engage in manipulation when derivable gains are appreciable. Apart from this, the gains or the factor of increments show the extent of susceptibility to manipulation and provide a measure of domination of manipulability among the indices. The magnitude of this gain for unanimity WVGs, as we shall see shortly, depends on the number of agents in the original game, the number of agents the annexer is able to annex, as well as the annexation costs. Example 5 illustrates a unanimity WVG where an annexer appears to achieve three times its original power annexing other agents.

Example 5. *Annexation Advantageous: Unanimity WVGs. Consider* $G = [\boldsymbol{7}, \boldsymbol{6}, \boldsymbol{9}, \boldsymbol{2}, \boldsymbol{5}, \boldsymbol{3}, \boldsymbol{1}, \boldsymbol{1}, \boldsymbol{8}, \boldsymbol{2}, \boldsymbol{2}, 8, 4, 9, 6; 73]$, *a unanimity WVG of* 15 *agents. The Deegan-Packel index of any agent in the game is* 0.0667. *Suppose the first agent with weight* 7, *alters G by annexing the next ten agents in the game. The new game* $G' = [\boldsymbol{46}, 8, 4, 9, 6; 73]$. *The annexer has improved its weight to* 46. *The Deegan-Packel index of the annexer in* G' *is* $\gamma_1(G') = 0.2000$. *The agent benefits from the annexation and increases its power by a factor of* 3.

Table 5: The annexer power in the original game $G = [\mathbf{7}, \mathbf{6}, \mathbf{9}, \mathbf{2}, \mathbf{5}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{2}, \mathbf{2}, 8, 4, 9, 6; 73]$, the altered game $G' = [\mathbf{46}, 8, 4, 9, 6; 73$, and the factor of increment for the three indices.

Power Index	G	G'	Factor
Shapley-Shubik	0.0667	0.2000	3.00
Banzhaf	0.0667	0.2000	3.00
Deegan-Packel	0.0667	0.2000	3.00

It appears that the annexer has achieved a gain of three times its original power while annexing other agents, but this is not true in its entirety. We provide the following arguments. Since the original and the altered games are unanimity, the power of all agents in each game is the same. While the annexer has improved its weight, and consequently its power by three times its original power, other agents that were not assimilated have also had their power increased by the same factor, even though their weights in the original and altered games remain the same. Clearly, these agents do not incur any cost like the annexer whose improved weight and power must have been achieved at annexation costs. The annexation costs reduce the benefits the agent thought it gained, making the annexer's benefit worse than the benefits of other agents not engaging in annexation. This weakens (Aziz and Paterson, 2009) result that for unanimity WVG and for both Shapley-Shubik and Banzhaf indices it is advantageous for a player to annex.

Now, suppose we assume that the annexer still accrues some gains even after the application of the annexation costs, then these gains are the same for the three indices. We see that the extents of susceptibility to manipulation among the three indices are the same. Hence, for any unanimity WVGs, the manipulability of any one index does not dominate the manipulability of other indices.

Finally, the generalization of the upper bound on the extent to which a strategic agent may gain with respect to games it manipulate in any unanimity WVGs follows from (Aziz and Paterson, 2009). For any unanimity WVG of n agents, the power of each agent is $\frac{1}{n}$. If a strategic agent annexes $k-1$ other agents, the power of the strategic agent as well as that of the other agents in the new game is $\frac{1}{n-k+1}$. Hence, the factor of increment for each agent is $\frac{n}{n-k+1}$. This factor of increment is the same for the three indices. When $k=1$, (i.e., the strategic agent is not annexing any other agent), then the factor of increment is 1, and this implies the same game we started with. Whereas, when $k=n$, the strategic agent is able to annex the remaining $n-1$ agents in the original game, then the factor of increment is n times the power of the agent in the original game. This is the upper bound on the extent to which a strategic agent may achieve while annexing other agents in any unanimity WVG. This bound holds for the three power indices.

5.2 Non Unanimity Weighted Voting Games

Manipulation by annexation and merging in the general case of WVGs is more interesting as it provides more complex and realistic scenarios that are not well-understood. As the structure of the WVGs changes due to annexation and merging, the number of winning coalitions as well as the minimal winning coalitions in the games also changes.

Consider a WVG G of I agents with quota q. If any agent $i \in I$ has weight $w_i \geq q$, then the agent will always win without forming coalitions with other agents. The more interesting games we consider are those for which $w_i < q$, and such that q satisfies the inequality $q < w(I) - m$, where m is at least the weight of exactly one of the agents in the game. When the grand coalition (i.e., a coalition of all the agents) emerges, it will always contain some agents that are not critical in the coalition. It is easy to see that all the winning coalitions in this type of games are non unanimity; hence, all the games here are non unanimity WVGs. In order to evaluate the behaviors of the indices for non unanimity WVGs, we conduct experiments to evaluate the effects of manipulation when a strategic agent annexes some other agents in the games or when some manipulators merge to form blocs using each of the three indices. The simulation environment and simulation results are discussed in Subsections 6.1 and 6.2, respectively.

6 EXPERIMENTS

This section provides detail descriptions of the simulation environment used for the conduct of experiments, and analysis of the experimental results used for the evaluation of the effects of manipulation via annexation and merging in non unanimity WVGs.

6.1 Simulation Environment

We perform experiments to evaluate the effects of manipulation via annexation and merging by agents using each of the three power indices. To facilitate comparison, we have 15 agents in each of the original WVGs. The weights of our agents in these games are chosen so that no weight is larger than ten. These weights are reflective of realistic voting procedures as the weights of agents in real votings are not too large (Bachrach and Elkind, 2008). When creating a new game, all agents are randomly assigned weights and the quota of the game is also generated to satisfy the inequality of non unanimity WVGs of Subsection 5.2. The least possible weight for any agent is one.

For the case of manipulation via annexation, we randomly generate WVGs and assume that only the first agent in the game is engaging in the manipulation, i.e., the annexer. Then, we determine the three power indices (i.e., Shapley-Shubik, Banzhaf, and Deegan-Packel power index) of this agent in the game. After this, we consider annexation of at least one agent in the game by the annexer, while the weights of other agents not annexed remain the same in the altered games. For a particular game, the annexer may annex $1 \leq i \leq 10$ other agents; we refer to

i as the *bloc size*. The bloc size is randomly generated for each game. The weight of the annexer in the new game is the sum of the weights of the agents it annexed plus the annexer's initial weight in the original game. We compute the new power index of the annexer in the altered games next. Now, we determine the factor of increment by which the annexer gains or loses in the manipulation for the corresponding bloc sizes i, in the range $1 \leq i \leq 10$.

We use the same procedure as described above for the case of manipulation via merging with the following modifications. Since merging requires coordinated action of the manipulators, we randomly select strategic agents among the agents in the WVGs to form the blocs of manipulators. The bloc size $2 \leq i \leq 10$, for mergng is also randomly generated for each game. The weight of a bloc in a new game is the sum of the weights of the assimilated agents in the bloc. The bloc participates in the new game as though a single agent. We compute the new power index of the bloc in the altered games next. Again, we determine the factor of increment by which the bloc gains or loses in the manipulation for the corresponding bloc sizes. Unlike in annexation, the power of the bloc is compared with the sum of the original powers of the individual agents in the bloc.

For our study, we generate $2,000$ original WVGs for various bloc sizes and allow manipulation by the annexer or the bloc of manipulators. For each game, we compute the factor of increment by which the annexer or the bloc gains or loses. Finally, we compute the average value of these factors of increment over all the games for each bloc size. We use $2,000$ WVGs in order to capture a variety of games that are representative of the non unanimity WVGs and to minimize the standard deviation from the true factors when we compute the average values. The average value of the factors of increment provides the extent of susceptibility to manipulation by each of the three indices. We estimate the domination of manipulability among the three indices by comparing their average factors of increment simultaneously in similar games.

6.2 Simulation Results

We present the results of our simulations. Experiments confirm the existence of advantageous annexation and merging for the non unanimity weighted voting games when agents engage in manipulation using the three indices. However, the extent to which agents gain varies with both annexation and merging, and among the indices.

Consider manipulation by annexation in non unanimity WVGs first. We provide a comparison of susceptibility to manipulation among the three indices by comparing the population of factors of increment attained by strategic agents in different games for each of the indices. A summary of susceptibility to manipulation among the three indices for $2,000$ WVGs is shown in Figure 1. The x-axis indicates the bloc sizes while the y-axis is the average factor of increment achieved by agents in the $2,000$ WVGs for corresponding bloc sizes.

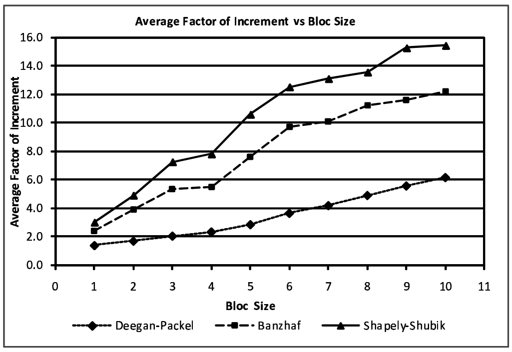

Figure 1: Susceptibility to Manipulation via Annexation among the Shapley-Shubik, Banzhaf, and Deegan-Packel indices for Non Unanimity WVGs.

The effect of manipulation via annexation is pronounced for the three power indices, as all the indices are highly susceptible to manipulation. However, the higher susceptibility of the Shapley-Shubik and Banzhaf indices than the Deegan-Packel index can be observed from Figure 1. While the average factor of increment for manipulation rapidly grows with the bloc sizes for the Shapley-Shubik and Banzhaf indices, that of the Deegan-Packel index grows more slowly. By the average factor of increment, the Shapley-Shubik index manipulability dominates that of Banzhaf index, which in turn dominates that of Deegan-Packel index. Also, there is a positive correlation between the average factor of increment and the bloc sizes for the three indices. The average factor of increment increases with the bloc sizes.

This analysis suggests that the Shapley-Shubik and Banzhaf power indices are more susceptible to manipulation via annexation than the Deegan-Packel power index. Since all the three power indices are susceptible to manipulation via annexation, this provides some motivation for strategic agents to generally engage in such manipulation for the non unanimity WVGs when they are being evaluated using any of the three power indices, and in particular, when the Shapley-Shubik index is employed.

Figure 2 provides similar results for the non unanimity WVGs when there are coordinated efforts among manipulators that culminate in merging. We

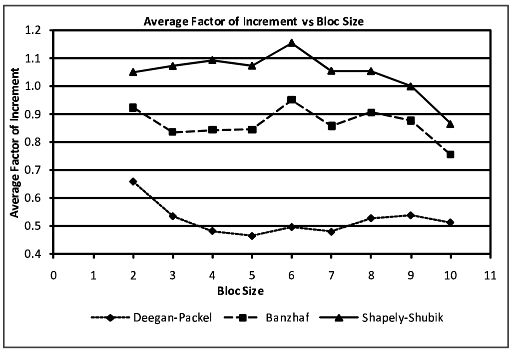

Figure 2: Susceptibility to Manipulation via Merging among the Shapley-Shubik, Banzhaf, and Deegan-Packel indices for Non Unanimity WVGs.

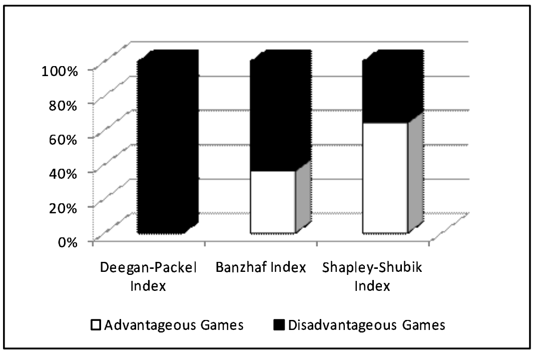

Figure 3: Percentage of Advantageous and Disadvantageous Games for Manipulation via Merging among the three indices for $2,000$ Non Unanimity WVGs.

again compare susceptibility to manipulation among the three power indices. Unlike manipulation via annexation, only the Shapley-Shubik index appears to be susceptible to manipulation for this type of game. Also, there appears not to be any correlation between the average factor of increment achieved by the bloc of manipulators and the bloc size for the three power indices. Thus, it is unclear to the would-be manipulators what bloc size would be advantageous or disadvantageous to the bloc, and to what extent.

It is easy to see from the trends of the three power indices in Figure 2, that, using the average factor of increment over the games we consider, the Shapley-Shubik index manipulability dominates that of the Banzhaf index, which in turn dominates that of the Deegan-Packel index. Another positive result that is observable from Figure 2 is that the highest average factor of increment for the three power indices is less than a factor of 1.2 as compared to a factor of 15, found for the Shapley-Shubik index, 12 for the Banzhaf index, and 6 for the Deegan-Packel index under the manipulation via annexation. See Figure 1. Again, examination of the $2,000$ WVGs reveals that many of the games are advantageous for Shapley-Shubik index, few for the Banzhaf index, and virtually none for the Deegan-Packel index. Figure 3 shows the percentage of advantageous and disadvantageous games for manipulation via merging among the three indices for the $2,000$ non unanimity WVGs. Even for the cases where the games are advantageous for the three indices, the factor of increment achieved by the blocs of manipulators are not very high, and in all cases are less than a factor of 2.

The analysis suggests that the the Shapley-Shubik index is more susceptible to manipulation via merging than the Banzhaf and Deegan-Packel power indices for non unanimity WVGs, even though the factor of increment is not high. Now, since only the Shapley-Shubik index is more susceptible to manipulations via merging, and also, since the factor by which the bloc of manipulators gains is very low, we suspect that this may provide less motivation for strategic agents to generally engage in manipulation via merging for the non unanimity WVGs when they are being evaluated using any of the three power indices, and in particular, when the Deegan-Packel index is employed.

7 CONCLUSIONS

We have considered the effects of manipulation by annexation and merging in weighted voting games focusing on the indices used in evaluating agents' power in such games. The following prominent power indices are used to evaluate the power of agents: Shapley-Shubik, Banzhaf, and the Deegan-Packel indices. We consider the extent to which strategic agents may gain by engaging in such manipulation and show how the susceptibility among the three indices compares for unanimity and non unanimity weighted voting games.

For unanimity weighted voting games of n agents, we show that apart from the fact that annexation always increases the power of other agents that are not annexed by the same factor of increment as the annexer achieved, the annexer also incurs annexation costs that reduce the benefit the agent thought it gained, making the annexer's benefit worse than the benefits of other agents not engaging in annexation. Also, for the three power indices, the manipulability of any one index does not dominate the manipulability of other indices for manipulation via annexation. Finally, the upper bound on the extent to which a strategic agent may gain while annexing other agents in the altered game is at most n times the power of the agent in the original game. This bound holds for the three power indices.

For non unanimity weighted voting games, we show that the games are less vulnerable to manipulation via merging, while they are extremely vulnerable to manipulation via annexation for the three power indices. Also, while the factor of increment from manipulation grows with bloc sizes for manipulation via annexation, there exists no correlation between the factor of increment and the bloc size for manipulation via merging. Finally, we show that the Shapley-Shubik index manipulability dominates that of the Banzhaf index, which in turn dominates that of the Deegan-Packel index for both manipulation via annexation and merging. Hence, the Shapley-Shubik index is more susceptible to manipulation via annexation and merging than the Banzhaf and Deegan-Packel indices, with Deegan-Packel index being the least susceptible among the indices.

We have some comments regarding these results. First, we found that our results of manipulation via annexation and merging for non unanimity weighted voting games are consistent with those of (Lasisi and Allan, 2010). They consider false name manipulation in weighted voting games. The manipulation allows an agent to have more power over the outcomes of the games by splitting into multiple names and distributing its weights across all associated names. They showed that for the non unanimity weighted voting games; the Deegan-Packel index is more susceptible to false name manipulation than the Banzhaf and Shapley-Shubik indices, with Shapley-Shubik index being the least susceptible among the three indices. The implication of this consistency is that a scenario where splitting by a strategic agent is disadvantageous corresponds to a scenario where it is advantageous for several strategic agents to merge.

Second, we have assumed throughout this paper that for the case of manipulation via merging, the assimilated agents in the bloc can easily distribute the gains from their collusion among themselves in a fair and stable way. Thus, making them agree to engage in the manipulation if it is profitable. This assumption is strong. Even at that, we see that all the three indices are less vulnerable to manipulation via merging.

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REFERENCES

Alonso-Meijide, J. M. and Bowles, C. (2005). Generating functions for coalitional power indices:an application to the imf. *Annals of Operations Research*, 137:21–44.

Aziz, H. and Paterson, M. (2009). False-name manipulations in weighted voting games: splitting, merging and annexation. In *Proceedings of the Intl. Joint Conf. on Autonomous Agents and Multiagent Systems*, pages 409–416, Budapest, Hungary.

Aziz, H., Paterson, M., and Leech, D. (2007). Combinatorial and computational aspects of multiple weighted voting games. *The Warwick Economics Research Paper Series (TWERPS) 823, University of Warwick, Department of Economics*.

Bachrach, Y. and Elkind, E. (2008). Divide and conquer: false-name manipulations in weighted voting games. In *7th Intl. Conf. on Autonomous Agents and Multiagent Systems*, pages 975–982, Estoril, Portugal.

Bachrach, Y., Markakis, E., Procaccia, A. D., Rosenschein, J. S., and Saberi, A. (2008). Approximating power indices. In *7th Intl. Conf. on Autonomous Agents and Multiagent Systems*, pages 943–950, Estoril, Portugal.

Deng, X. and Papadimitriou, C. H. (1994). On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19(2):257–266.

Garey, M. and Johnson, D. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H. Freeman, San Fransisco.

Kirsch, W. (2007). On penrose’s squareroot law and beyond. *Homo Oeconomicus*, 24(3,4):357–380.

Kirsch, W. and Langner, J. (2010). Power indices and minimal winning coalitions. *Social Choice and Welfare*, 34(1):33–46.

Laruelle, A. (1999). On the choice of a power index. *Instituto Valenciano de Investigaciones Economicas*, 2103:99–10.

Laruelle, A. and Valenciano, F. (2005). Assessing success and decisiveness in voting situations. *Social Choice and Welfare*, 24(1):171–197.

Lasisi, R. O. and Allan, V. H. (2010). False name manipulations in weighted voting games: Susceptibility of power indices. In *proceedings of the 13th Workshop on Trust in Agents Societies of the Autonomous Agents and Multiagents Systems Conference*, pages 139–150, Toronto, Canada.

Leech, D. (2002). Voting power in the governance of the international monetary fund. *Annals of Operations Research*, 109(1):375–397.

Levchenkova, L. and Levchenkov, V. (2002). A power index in weighted voting systems. *Journal of Computational Mathematics and Modelling*, 13(4):375–392.

Machover, M. and Felsenthal, D. S. (2002). Annexation and alliances: When are blocs advantageous a priori. *Social Choice and Welfare*, 19(2):295–312.

Matsui, T. and Matsui, Y. (2000). A survey of algorithms for calculating power indices of weighted majority games. *Journal of the Operations Research Society of Japan*, 43(1).